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On an exponential-type dynamical system: $\dot{x}_k = (-1/2)[\exp(x_{k-1} - x_k) + \exp(x_k - x_{k+1})]$

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Abstract. We integrate the system of non-linear differential equations $\dot{x}_k = (-1/2)[\exp(x_{k-1} - x_k) + \exp(x_k - x_{k+1})]$ and study the behaviour of a system of infinitely many mass points on the line governed by these differential equations. The results are compared with those for the corresponding finite system.

1. Introduction

In close connection with the system of non-linear differential equations

$$\dot{a}_k = a_k(a_{k+1}^2 - a_{k-1}^2) \qquad k = 1, \ldots, n-1;\ a_0 = a_n = 0,\ n = 2\nu \tag{1.1}$$

Moser [1] studied a system composed of n particles obeying

$$\dot{x}_k = (-1/2)[\exp(x_{k-1} - x_k) + \exp(x_k - x_{k+1})] \tag{1.2}$$

($k = 1, \ldots, n$; $x_0 = -\infty$, $x_{n+1} = +\infty$, $n = 2\nu$), where x_k denotes the position of the kth particle. With $a_k = (1/2)\exp[(x_k - x_{k+1})/2]$, (1.1) follows from (1.2). It also follows from (1.2) that

$$\ddot{x}_k = (1/4)[\exp(x_{k-2} - x_k) - \exp(x_k - x_{k+2})] = -\partial U/\partial x_k \tag{1.3}$$

with

$$U = \frac{1}{4} \sum_{1 \leqslant k \leqslant n-2} \exp(x_k - x_{k+2}).$$

From the viewpoint of practical application, studying such a system may seem to bear relatively little significance. But theoretically it is still interesting, since it sheds some light on the nature of exponential-type dynamical systems [2, 3].

In this system of n mass points, the force induced between the kth and the $(k+2)$th particles is such that it makes the kth one move left relative to the $(k+2)$th. And since each of the end particles ($k = 1, 2, n-1, n$) is connected to only one other particle, all the particles asymptotically ($t \to \pm\infty$) behave as though free, each travelling with a certain constant velocity, i.e.

$$x_k(t) \sim \alpha_k^{\pm} t + \beta_k^{\pm} \qquad t \to \pm\infty. \tag{1.4}$$

In the scattering problem, one is asked to determine the relation among these asymptotic velocities $\alpha_k^{\pm}$ and phases $\beta_k^{\pm}$ ($k = 1, \ldots, n$). Without knowing any detailed

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structure of the solution, and using only rudimentary properties of its rational character, Moser [1] derived the following relations:

$$\alpha_{2l}^{+}=\alpha_{2l-1}^{+}=\alpha_{n-2l+2}^{-}=\alpha_{n-2l+1}^{-}=-2\lambda_l^2 \tag{1.5}$$

$$\beta_{2l-1}^{+}-\beta_{2l}^{+}=\beta_{n-2l+1}^{-}-\beta_{n-2l+2}^{-}=\ln(-2\alpha_{2l}^{+}) \tag{1.6}$$

$$\beta_{2l}^{+}-\beta_{n-2l+2}^{-}=\sum_{\nu-l+2\leqslant k\leqslant\nu}[-\ln 4(\alpha_{2k}^{-}-\alpha_{n-2l+2}^{-})^2]-\sum_{1\leqslant k\leqslant \nu-l}[-\ln 4(\alpha_{2k}^{-}-\alpha_{n-2l+2}^{-})^2] \tag{1.7}$$

where $l=1,\ldots,\nu$ and λ_k ($\lambda_n<\ldots<\lambda_2<\lambda_1$, $\lambda_l=-\lambda_{n-l+1}$ are the eigenvalues of the relevant $(n\times n)$ Jacobi matrix.

In these relations, (1.5) implies that the particles asymptotically travel in pairs, the lth pair being composed of the $(2l-1)$th and the $2l$th particles, and that the asymptotic velocities are exchanged between the lth and the $(\nu-l+1)$th pairs. The relation (1.6) concerns the asymptotic distances of two particles constituting a pair. Finally (1.7) admits the interpretation that in the scattering of these particles, the pairs behave as if they interacted pairwise at a time.

In [4], under the condition on the initial value of a_k that the corresponding Jacobi matrix represents a compact Hermitian operator in the Hilbert space l_2, the system of differential equations (1.1) for $n=\infty$ was studied. It turned out that the results can be regarded as a natural extension of those for the finite system. It is then natural to try to investigate the system (1.2) for $n=\infty$, and to compare the results with those for the finite system, as is the theme of this paper.

2. Integration and behaviour of the infinite system

As we shall derive below, this infinite system still asymptotically behaves like the one composed of free particles, each travelling with a certain constant velocity. Intuitively this may be considered as resulting from the existence of the end particles (x_1 and x_2). Thus the forces acting on the end particles are such that they make x_1 and x_2, left of x_3 and x_4 respectively, asymptotically move freely. Consequently the particles x_3 and x_4, and then x_5 and x_6 etc play the role of the end particles, and tend to move freely.

Under the restriction on the initial value of a_k that the Jacobi matrix L defined by

$$(L)_{k,k+1}=(L)_{k+1,k}=a_k(0) \qquad (L)_{j,k}=0 \quad \text{otherwise} \tag{2.1}$$

stands for a compact Hermitian operator in the Hilbert space l_2, the solution of (1.1) for $n=\infty$ is given by [4]

$$A_k(\equiv a_k^2)=\mathscr{P}_{k-2}\mathscr{P}_{k+1}/\mathscr{P}_{k-1}\mathscr{P}_k \qquad k=1,2,\ldots \tag{2.2}$$

with

$$\mathscr{P}_{2l-1}(t)=\sum_{\{k_1,\ldots,k_l\}}R_{k_1}(t)\ldots R_{k_l}(t)\Delta(\lambda_{k_1}^2,\ldots,\lambda_{k_l}^2)^2$$

$$\mathscr{P}_{2l}=\sum_{\{k_1,\ldots,k_l\}}R_{k_1}(t)\ldots R_{k_l}(t)\lambda_{k_1}^2\ldots\lambda_{k_l}^2\Delta(\lambda_{k_1}^2,\ldots,\lambda_{k_l}^2)^2 \tag{2.3}$$

$$\Delta(\lambda_{k_1}^2,\ldots,\lambda_{k_l}^2)\equiv\prod_{1\leqslant i<j\leqslant l}(\lambda_{k_j}^2-\lambda_{k_i}^2) \qquad \Delta(\lambda_{k_1}^2)\equiv 1$$

$$(\mathscr{P}_0=\mathscr{P}_{-1}=1,\ \mathscr{P}_k\neq 0\ (k=1,2,\ldots))$$

where $\pm\lambda_k$ $(-\lambda_1<-\lambda_2<\ldots<0<\ldots<\lambda_2<\lambda_1)$ are the eigenvalues of L, $R_k(0)/2\,\Sigma_{1\leqslant k<\infty}\,R_k(0)$ ($R_k(0)>0$; $k=1,2,\ldots$) is the discontinuity of a certain purely discontinuous distribution function $\rho(\lambda)$ at $\lambda = \pm\lambda_k$, $R_k(t) = R_k(0)\exp(2\lambda_k^2 t)$ and $\Sigma_{\{k_1,\ldots,k_l\}}$ denotes summation over all combinations of $k_1, \ldots, k_l$ ($\in \mathbb{N}$).

With the aid of this result, we recognise that

$$x_1(t) = x_1(0) + \ln[\mathscr{P}_1(0)/\mathscr{P}_1(t)]$$

since

$$\begin{aligned}\dot{x}_1 &= (-1/2)\exp(x_1 - x_2) = -2A_1 = -2\mathscr{P}_2/\mathscr{P}_1 = -\dot{\mathscr{P}}_1/\mathscr{P}_1\\ &= -\mathrm{d}(\ln \mathscr{P}_1)/\mathrm{d}t.\end{aligned}$$

So

$$\begin{aligned}x_k(t) &= x_1(t) - \sum_{1\leq j\leq k-1}[x_j(t) - x_{j+1}(t)]\\ &= x_1(t) - \sum_{1\leq j\leq k-1}\ln[4A_j(t)]\\ &= x_1(0) + \ln\left[\mathscr{P}_1(0)/\mathscr{P}_1(t)\prod_{1\leq j\leq k-1}4A_j(t)\right]\\ &= x_1(0) + \ln[\mathscr{P}_1(0)\mathscr{P}_{k-2}(t)/4^{k-1}P_k(t)] \qquad k = 1, 2, \ldots.\end{aligned} \tag{2.4}$$

Now, by virtue of the behaviour of $\mathscr{P}_k$ (see [4]) we have as $t \to +\infty$

$$\ln[\mathscr{P}_{2l-3}/\mathscr{P}_{2l-1}] \sim -2\lambda_l^2 t - \ln R_l(0) - \sum_{1\leq k\leq l-1}\ln(\lambda_l^2 - \lambda_k^2)^2$$

$$\ln[\mathscr{P}_{2l-2}/\mathscr{P}_{2l}] \sim -2\lambda_l^2 t - \ln[R_l(0)\lambda_l^2] - \sum_{1\leq k\leq l-1}\ln(\lambda_l^2 - \lambda_k^2)^2$$

so

$$\alpha_{2l-1}^+ = \alpha_{2l}^+ = -2\lambda_l^2 \tag{2.5}$$

$$\beta_{2l-1}^+ = x_1(0) + \ln[\mathscr{P}_1(0)/4^{2l-2}R_l(0)] - \sum_{1\leq k\leq l-1}\ln(\lambda_l^2 - \lambda_k^2)^2 \tag{2.6}$$

$$\beta_{2l}^+ = x_1(0) + \ln[\mathscr{P}_1(0)/4^{2l-1}R_l(0)\lambda_l^2] - \sum_{1\leq k\leq l-1}\ln(\lambda_l^2 - \lambda_k^2)^2 \tag{2.7}$$

and

$$\beta_{2l-1}^+ - \beta_{2l}^+ = \ln[4\lambda_l^2]. \tag{2.8}$$

On the other hand as $t \to -\infty$, since all the $A_k(t)$ ($k = 1, 2, \ldots$) and $P_1(t) = \Sigma_{1\leq k<\infty} R_k(0)\exp(2\lambda_k^2 t)$ tend to zero [4], it follows that

$$\begin{aligned}x_k(t) &= x_1(0) + \ln\left[\mathscr{P}_1(0)/\mathscr{P}_1(t)\prod_{1\leq j\leq k-1}4A_j(t)\right]\\ &\to +\infty \qquad k = 1, 2, \ldots\end{aligned} \tag{2.9}$$

and

$$x_{k+1}(t) - x_k(t) = -\ln[4A_k(t)] \to +\infty \qquad k = 1, 2, \ldots \tag{2.10}$$

Further, from (1.2) and (2.10), it also follows that

$$\dot{x}_k(t) \to -0 \qquad k = 1, 2, \ldots \tag{2.11}$$

i.e.

$$\alpha_k^- = 0 \qquad \beta_k^- = +\infty \qquad k = 1, 2, \ldots. \tag{2.12}$$

So that as $t \to -\infty$, the particles asymptotically stay still at infinity, the kth particle to the left of the $(k+1)$th with the separation $x_{k+1} - x_k = \infty$.

3. Comparison with the finite system

To compare our results for the infinite system with those for the finite system, it is desirable to know the phases $\beta_k^\pm$ of the finite system rather than their differences ((1.6) and (1.7)), which we shall derive from the solution for the finite system.

The solution for the finite system (1.2) is afforded by the expressions (2.2) for $k = 1, \ldots, n-1$ and (2.3), where λ_k ($\lambda_n < \lambda_{n-1} < \ldots < \lambda_2 < \lambda_1$; $\lambda_k = -\lambda_{n-k+1}$ ($k = 1, \ldots, \nu$)) are the eigenvalues of the finite $(n \times n)$ Jacobi matrix defined in terms of the initial value of a_k ($k = 1, \ldots, n-1$) by (2.1), and $\Sigma_{\{k_1,\ldots,k_l\}}$ denotes summation over all combinations of $k_1, \ldots, k_l$ taken from the set $\{1, 2, \ldots, \nu\}$ (the number of these combinations is ${}_\nu C_l$).

The asymptotic behaviour of $\mathscr{P}_k$ as $t \to +\infty$ is formally the same as that for the infinite system, so the results (2.5) ~ (2.8) with $l = 1, \ldots, \nu$ hold also for the finite system.

The behaviour of $\mathscr{P}_k$ as $t \to -\infty$ turns out to be

$$\mathscr{P}_{2l-1}(t \to -\infty) \sim R_\nu(0) R_{\nu-1}(0) \ldots R_{\nu-l+1}(0) \Delta(\lambda_\nu^2, \lambda_{\nu-1}^2, \ldots, \lambda_{\nu-l+1}^2)^2 \times \exp[2(\lambda_\nu^2 + \lambda_{\nu-1}^2 + \ldots + \lambda_{\nu-l+1}^2)t]$$

$$\mathscr{P}_{2l}(t \to -\infty) \sim R_\nu(0) R_{\nu-1}(0) \ldots R_{\nu-l+1}(0) (\lambda_\nu^2 \lambda_{\nu-1}^2 \ldots \lambda_{\nu-l+1}^2) \Delta(\lambda_\nu^2, \lambda_{\nu-1}^2, \ldots, \lambda_{\nu-l+1}^2)^2 \times \exp[2(\lambda_\nu^2 + \lambda_{\nu-1}^2 + \ldots + \lambda_{\nu-l+1}^2)t] \qquad l = 1, 2, \ldots, \nu.$$

So

$$\mathscr{P}_{2l-3}/\mathscr{P}_{2l-1}(t \to -\infty) \sim \left(R_{\nu-l+1}(0) \prod_{\nu-l+2 \leqslant k \leqslant \nu} (\lambda_{\nu-l+1}^2 - \lambda_k^2)^2 \right)^{-1} \exp(-2\lambda_{\nu-l+1}^2 t) \tag{3.1a}$$

$$\mathscr{P}_{2l-2}/\mathscr{P}_{2l}(t \to -\infty) \sim \left(R_{\nu-l+1}(0) \lambda_{\nu-l+1}^2 \prod_{\nu-l+2 \leqslant k \leqslant \nu} (\lambda_{\nu-l+1}^2 - \lambda_k^2)^2 \right)^{-1} \exp(-2\lambda_{\nu-l+1}^2 t). \tag{3.1b}$$

From (2.4) and (3.1), we recognise that

$$\alpha_{2l-1}^- = \alpha_{2l}^- = -2\lambda_{\nu-l+1}^2 \tag{3.2}$$

$$\beta_{2l-1}^- = x_1(0) + \ln[\mathscr{P}_1(0)/4^{2l-2} R_{\nu-l+1}(0)] - \sum_{\nu-l+2 \leqslant k \leqslant \nu} \ln(\lambda_{\nu-l+1}^2 - \lambda_k^2)^2 \tag{3.3a}$$

$$\beta_{2l}^- = x_1(0) + \ln[\mathscr{P}_1(0)/4^{2l-1} \lambda_{\nu-l+1}^2 R_{\nu-l+1}(0)] - \sum_{\nu-l+2 \leqslant k \leqslant \nu} \ln(\lambda_{\nu-l+1}^2 - \lambda_k^2)^2 \tag{3.3b}$$

and

$$\beta_{2l-1}^- - \beta_{2l}^- = \ln(4\lambda_{\nu-l+1}^2) \qquad l = 1, 2, \ldots, \nu. \tag{3.4}$$

In these relations, (3.2) and (3.4) are included in (1.5) and (1.6) respectively. The relation (1.7) results from (2.7) and (3.3b).

Now as $n(=2\nu) \to \infty$, since $\lambda_{\nu-l+1} \to 0$ ($l = 1, 2, \ldots$) it follows from (3.2) that $\alpha_k^- \to -0$ ($k = 1, 2, \ldots$) (cf (2.11)). Similarly from (3.3) we recognise that $\beta_k^- \to +\infty$ ($k = 1, 2, \ldots$) (cf (2.12)), so $x_k(t) \to +\infty$ ($t \to -\infty$; $k = 1, 2, \ldots$) (cf (2.9)).

As for the distances $x_{k+1}(t)-x_k(t)$ (cf (2.10)), since $\alpha_{2l+1}^- - \alpha_{2l}^- = 2(\lambda_{\nu-l+1}^2-\lambda_{\nu-l}^2)<0$, we see that for any n $(=2\nu<\infty)$

$$x_{2l+1}(t)-x_{2l}(t)\sim(\alpha_{2l+1}^- - \alpha_{2l}^-)t+(\beta_{2l+1}^- - \beta_{2l}^-)$$
$$\to+\infty \qquad t\to-\infty$$

while for $k=2l-1$

$$x_{2l}(t)-x_{2l-1}(t)\sim-\ln(4\lambda_{\nu-l+1}^2) \qquad t\to-\infty$$
$$\to+\infty \qquad n\to\infty.$$

So, as far as the relations (3.2)-(3.4) are concerned, each of them suitably translates as $n\to\infty$ to the corresponding relation for the infinite system. For the relation (1.7), it seems that it has no longer any appropriate limit as $n\to\infty$.

4. Conclusion

From these considerations, we can describe the behaviour of the infinite system as follows.

The configuration of the system in the past $(t\to-\infty)$ is characterised by (2.9), (2.10) and (2.11). As time evolves, it changes to become the one prescribed by the initial condition which imposes (if we adopt the condition that L is an operator of Hilbert-Schmidt class, which is stronger than the requirement of compactness) that

$$\sum_{1\le j,k<\infty}[(L)_{j,k}]^2=2\sum_{1\le k<\infty}a_k(0)^2$$
$$=\frac{1}{2}\sum_{1\le k<\infty}\exp[x_k(0)-x_{k+1}(0)]$$
$$<\infty.$$

Thus for large k $(k\to\infty)$, the relations (2.9)-(2.11) still hold at $t=0$.

The motion of the system as $t\to+\infty$ is characterised by (2.5)-(2.8), from which it follows that

$$x_k(t)\to-\infty \qquad t\to+\infty;\ k=1,2,\dots$$
$$x_{2l+1}(t)-x_{2l}(t)\to+\infty \qquad t\to+\infty;\ l=1,2,\dots$$
$$x_{2l}(t)-x_{2l-1}(t)\sim-\ln(4\lambda_l^2) \qquad t\to+\infty$$
$$\to+\infty \qquad l\to\infty$$

and

$$\alpha_k^+\to-0 \qquad k\to\infty.$$

As time evolves from $t=-\infty$ to $t=+\infty$, all the particles are moving from $x=+\infty$ to $x=-\infty$. Focusing our attention on those $x_k(t)$ with sufficiently large k $(k\to\infty)$, they continue to satisfy (2.10) and (2.11) at $t=-\infty$, 0 and $+\infty$.

It is tempting to consider that even in the infinite system the asymptotic velocities are exchanged between pairs of particles, as is indeed the case in the following sense. From (2.5), the asymptotic velocities as $t\to+\infty$ are such that

$$\alpha_2^+<\alpha_4^+<\dots<\alpha_{2l}^+<\dots<0$$

while as $t \to -\infty$, in addition to (2.11), we shall now show that

$$\dot{x}_{2l}(t) - \dot{x}_{2(l+1)}(t) \to +0 \qquad l = 1, 2, \ldots$$

i.e.

$$\ldots < \dot{x}_{2l} < \ldots < \dot{x}_4 < \dot{x}_2 < 0.$$

To show this, we notice that since $A_k(>0) \to +0(t \to -\infty)$,

$$\dot{A}_k / A_k = 2(A_{k+1} - A_{k-1}) \to +0 \qquad t \to -\infty;\ k = 1, 2, \ldots.$$

Then

$$\begin{aligned} \dot{x}_{2l} - \dot{x}_{2(l+1)} &= 2(A_{2l+1} + A_{2l+2} - A_{2l-1} - A_{2l}) \\ &= \dot{A}_{2l}/A_{2l} + \dot{A}_{2l+1}/A_{2l+1} \\ &\to +0 \qquad t \to -\infty\dagger. \end{aligned}$$

Finally, although we have not been successful in doing so, it is hoped to find a relation which corresponds to (1.7) and characterises the behaviour of the infinite system.

References

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† Similarly it can also be shown that $\dot{x}_k - \dot{x}_{k+1} \to +0$ $(t \to -\infty;\ k = 1, 2, \ldots)$.